

Scattering matrix for a general $gl(2)$ spin chain

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Abstract

We study the general L_0 -regular $gl(2)$ spin chain, i.e. a chain where the sites $\{i, i + L_0, i + 2L_0, \dots\}$ carry the same arbitrary representation (spin) of $gl(2)$. The basic example of such chain is obtained for $L_0 = 2$, where we recover the alternating spin chain.

Firstly, we review different known results about their integrability and their spectrum. Secondly, we give an interpretation in terms of particles and conjecture the scattering matrix between them.

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1 Introduction

The one-dimensional Heisenberg spin chain [1] is among the few many-body quantum systems for which one can compute exactly some physical quantities (spectrum, correlation functions, ...). This model was solved for the first time in the seminal paper of H. Bethe [2] where he succeeded to map the study of the spectrum to the resolution of transcendental equations, now called Bethe equations. Subsequently, the Quantum Inverse Scattering Method (QISM) has been introduced (see [3, 4] and for reviews [5, 6]) based on solutions of Yang-Baxter equation [7, 8]. This approach is very fruitful and, for example, provides a whole class of integrable spin chains associated to generic algebras (Yangians or quantum groups based on classical Lie algebras or superalgebras). Focusing on spin chains based on $gl(2)$ only, one can use the QISM approach to define and study e.g.: higher spin chains [9, 10], spin chain with impurities [11, 12] or alternating spin chain [13, 14, 15]. In these types of chains the spins are no more $1/2$, but can be arbitrary, a situation that is nowadays relevant for condensed matter experiments, where quasi-one-dimensional spin chains with different spins are studied, see e.g. [16].

It is thus natural to wonder whether a spin chain containing arbitrary spins can be studied through QISM approach. A first and immediate problem in such studies comes from the thermodynamical limit (when L , the number of sites, tends to infinity). Such limit is needed for a comparison with physical model, and it is rather obvious that one needs some regularity in the spin content of the chain so as to be able to compute relevant quantities while taking the limit $L \rightarrow \infty$.

The L_0 -regular spin chains (i.e. spin chains with a repeated motif containing L_0 spins in arbitrary representations) have been introduced [17, 18] to remedy this objection while keeping enough freedom to encompass most of the known cases. For instance, homogeneous spin chains (for any spin s) correspond to $L_0 = 1$, and one recovers the alternating spin chain for $L_0 = 2$. They also allow to define an integrable model [18] for spin chains with periodic array of impurities (while the original ones [19] were not integrable). L_0 -regular spin chains have been also studied using QISM approach in [17, 20, 18, 21].

In the antiferromagnetic regime, these models may be seen as lattice versions of a corresponding integrable relativistic quantum field theories. This link is very useful since it allows one to compare both models for which numerous exact results are known. From spin chains side, by solving the Bethe equations in the thermodynamical limit, it may be possible to compute exactly the scattering matrix between the excitations. It allows one to obtain indication on the underlying field theory and, then, on the long-distance physics. This program has been followed for different types of spin chain: Heisenberg model [22], higher spin chains [23, 24] and alternating spin $(1/2, 1)$ chain [25, 26]. In this paper, we tackle the problem to compute the scattering matrix for the general L_0 -regular spin chains based on $gl(2)$. After recalling some results about their spectrum, we give an interpretation of the excitations in terms of particles and conjecture an explicit form for the scattering matrix.

The outline of this paper is as follows. In section 2, we give the notations used throughout the paper. Then, we recall, in section 3, well-known results about the integrability of the general spin chains using the transfer matrices constructed from rational solutions of the Yang-Baxter equation. We also link these transfer matrices with the shift operator as well as the Hamiltonian of the mod-

els. Section 4 is devoted to the computation of the Bethe equations and their study in the string hypothesis. The two following sections give some results on the spectrum: the energy of the antiferromagnetic vacuum is given in section 5 and the dispersion law for the first excited states is established in section 6. Then, we propose, in section 7, a conjecture for the scattering matrix between these excited states. Finally, in section 8, we conclude on open problems.

2 Notation

gl(2) Lie algebra We introduced the spin s representation of $gl(2)$ given explicitly by

$$\begin{aligned}\pi_s(e_3) &= \sum_{n=1}^{2s+1} (s+1-n) E_{nn}^{(s)} \quad ; \quad \pi_s(e_+) = \sum_{n=1}^{2s} \sqrt{n(2s+1-n)} E_{n,n+1}^{(s)} \\ \pi_s(e_-) &= \sum_{n=1}^{2s} \sqrt{n(2s+1-n)} E_{n+1,n}^{(s)} \quad ; \quad \pi_s(e_0) = \sum_{n=1}^{2s+1} E_{nn}^{(s)} = \mathbb{I}_{2s+1}\end{aligned}\tag{2.1}$$

where $E_{nm}^{(s)}$ is a $(2s+1) \times (2s+1)$ matrix with 1 in the entry (n, m) and 0 otherwise. The spin s representation of $su(2)$ embedded in $gl(2)$ is generated by $\{\pi_s(e_3), \pi_s(e_+), \pi_s(e_-)\}$.

Set of representations We study a periodic $gl(2)$ spin chain of L sites with the spin s_i representation on the site i . To be able to take the thermodynamical limit, we restrict ourselves to the case of L_0 -regular spin chain (i.e. $s_i = s_{i+L_0}$). In this case, the length L of the chain must be chosen such that L/L_0 be an integer. We introduce the ordered set $\mathcal{S} = \{\bar{s}_1, \bar{s}_2, \dots, \bar{s}_{\mathcal{L}} \mid \bar{s}_j < \bar{s}_{j+1}\}$ of the different values of the spins s_i ($1 \leq i \leq L_0$) present in the spin chain. We denote by $L_{\bar{s}_j}$ the number of times \bar{s}_j appears in the sequence s_1, s_2, \dots, s_{L_0} which allows us to define the density of the spin \bar{s}_j in the chain by

$$\rho_{\bar{s}_j} = L_{\bar{s}_j}/L_0.\tag{2.2}$$

We get $\rho_{\bar{s}_1} + \dots + \rho_{\bar{s}_{\mathcal{L}}} = 1$. For convenience, we use the conventions $\bar{s}_0 = 0$ and $\bar{s}_{\mathcal{L}+1} = \infty$.

We will also need to consider the spins which are not present in the chain, so that we introduce the following sets, for $i = 0, 1, \dots, \mathcal{L}$,

$$\mathcal{R}_{\bar{s}_i} = \left\{ \bar{s}_i + \frac{1}{2}, \bar{s}_i + 1, \dots, \bar{s}_{i+1} - \frac{1}{2} \right\} =]\bar{s}_i, \bar{s}_{i+1}[\cap \frac{1}{2}\mathbb{Z}.\tag{2.3}$$

We define $\mathcal{R} = \bigcup_{i=0}^{\mathcal{L}} \mathcal{R}_{\bar{s}_i} = \frac{1}{2}\mathbb{Z}_{>0} \setminus \mathcal{S}$ which is the set of all the representations not used to construct the spin chain.

Finally, to make lighter the formulas, we will use sometimes ρ_j (resp. \mathcal{R}_j) instead of $\rho_{\bar{s}_j}$ (resp. $\mathcal{R}_{\bar{s}_j}$). For instance, $\mathcal{R}_0 = \{\frac{1}{2}, 1, \dots, \bar{s}_1 - \frac{1}{2}\}$ and $\mathcal{R}_{\mathcal{L}} = \{\bar{s}_{\mathcal{L}} + \frac{1}{2}, \bar{s}_{\mathcal{L}} + 1, \dots, \infty\}$.

Elementary functions In the whole paper, essentially two functions as well as their logarithm, their derivative and their Fourier transform are necessary to construct all the other ones. We use the following definition for the Fourier transform

$$\hat{f}(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ip\lambda} f(\lambda) d\lambda . \quad (2.4)$$

We encompass in figure 1 their explicit form and their relations, for $\hbar > 0$ and $0 < r < \frac{\pi}{\hbar}$.

$$\begin{array}{ccc}
G_r^{(\hbar)}(\lambda) = \frac{\sinh\left(\hbar\left(-\lambda - \frac{ir}{2}\right)\right)}{\sinh\left(\hbar\left(\lambda - \frac{ir}{2}\right)\right)} & \xrightarrow{\lim_{\hbar \rightarrow 0}} & -e_r(-\lambda) = \frac{-\lambda - \frac{ir}{2}}{\lambda - \frac{ir}{2}} \\
\downarrow & & \downarrow \quad i \ln(.) \\
\Gamma_r^{(\hbar)}(\lambda) = 2 \arctan\left(\frac{\tanh(\hbar\lambda)}{\tan\left(\frac{\hbar r}{2}\right)}\right) & & \varphi_r(\lambda) = 2 \arctan\left(\frac{2\lambda}{r}\right) \\
\downarrow & & \downarrow \quad \text{Derivative w.r.t. } \lambda \\
\gamma_r^{(\hbar)}(\lambda) = \frac{2\hbar \sin(\hbar r)}{\cosh(2\hbar\lambda) - \cos(\hbar r)} & & \frac{4r}{4\lambda^2 + r^2} \\
\downarrow & & \downarrow \quad \text{Fourier transform} \\
\hat{\gamma}_r^{(\hbar)}(p) = \frac{\sinh\left[\frac{p}{2}\left(\frac{\pi}{\hbar} - r\right)\right]}{\sinh\left(\frac{p\pi}{2\hbar}\right)} & & \exp\left(-\frac{r|p|}{2}\right)
\end{array}$$

Figure 1: Relations between the elementary functions used in the paper

We extend these definitions to degenerate cases in the following way. For $r = 0$:

$$G_0^{(\hbar)}(\lambda) = -1 \quad ; \quad \Gamma_0^{(\hbar)}(\lambda) = 0 \quad ; \quad \gamma_0^{(\hbar)}(\lambda) = 2\pi\delta(\lambda) \quad \text{and} \quad \hat{\gamma}_0^{(\hbar)}(p) = 1 , \quad (2.5)$$

and, when $r = \frac{\pi}{\hbar}$,

$$G_{\pi/\hbar}^{(\hbar)}(\lambda) = 1 \quad ; \quad \Gamma_{\pi/\hbar}^{(\hbar)}(\lambda) = 0 \quad ; \quad \gamma_{\pi/\hbar}^{(\hbar)}(\lambda) = 0 \quad \text{and} \quad \hat{\gamma}_{\pi/\hbar}^{(\hbar)}(p) = 0 . \quad (2.6)$$

We need also the more involved following functions defined, for $0 < r < \frac{\pi}{\hbar}$, by

$$\hat{\kappa}_r^{(\hbar)}(p) = \frac{\hat{\gamma}_r^{(\hbar)}(p)}{2 \cosh\left(\frac{p}{2}\right)} , \quad (2.7)$$

and

$$\mathcal{K}_r^{(\hbar)}(\lambda) = \exp \int_{-\infty}^{\infty} dp \frac{e^{-ip\lambda}}{p} \hat{\kappa}_r^{(\hbar)}(p) = \exp -i \int_0^{\infty} dp \frac{\sin(p\lambda)}{p} \frac{\sinh\left(\frac{p}{2}\left(\frac{\pi}{\hbar} - r\right)\right)}{\cosh\left(\frac{p}{2}\right) \sinh\left(\frac{p\pi}{2\hbar}\right)} . \quad (2.8)$$

We extend also the previous definitions to the cases $r = 0$ and $r = \pi/\hbar$ using the conventions:

$$\mathcal{K}_0^{(\hbar)}(\lambda) = -i \coth \left(\frac{\pi}{2} \left(\lambda - \frac{i}{2} \right) \right) \quad \text{and} \quad \mathcal{K}_{\pi/\hbar}^{(\hbar)}(\lambda) = 1. \quad (2.9)$$

The limit $\hbar \rightarrow 0$ of $\mathcal{K}_r^{(\hbar)}(\lambda)$ can be computed and we get

$$\mathcal{K}_r^{(0)}(\lambda) = \frac{\Gamma(-\frac{i\lambda}{2} + \frac{r+3}{4})\Gamma(\frac{i\lambda}{2} + \frac{r+1}{4})}{\Gamma(\frac{i\lambda}{2} + \frac{r+3}{4})\Gamma(-\frac{i\lambda}{2} + \frac{r+1}{4})}. \quad (2.10)$$

The limits at $\pm\infty$ will be also used in the following, for $0 \leq r \leq \frac{\pi}{\hbar}$,

$$\lim_{\lambda \rightarrow \pm\infty} G_r^{(\hbar)}(\lambda) = \exp(\mp i(\pi - \hbar r)) \quad \text{and} \quad \lim_{\lambda \rightarrow \pm\infty} \mathcal{K}_r^{(\hbar)}(\lambda) = \exp(\mp \frac{i}{2}(\pi - \hbar r)). \quad (2.11)$$

3 Integrable Hamiltonians

3.1 Monodromy and transfer matrices

We will need monodromy matrices of different types, depending on the auxiliary space representation. Indeed, for $i = 1, \dots, \mathcal{L}$, we define the monodromy matrix with auxiliary space in the spin \bar{s}_i representation as

$$T_0^{(\bar{s}_i)}(u) = \overrightarrow{\prod}_{0 \leq p < L/L_0} R_{0,1+pL_0}^{(\bar{s}_i, s_1)}(u) \dots R_{0,(p+1)L_0}^{(\bar{s}_i, s_{L_0})}(u) \quad (3.1)$$

where the product is ordered $\overrightarrow{\prod}_{0 \leq i < L/L_0} X_{1+i} = X_1 X_2 \dots X_{L/L_0}$. $R_{0,j}^{(s_i, s_j)}(u)$ may be obtained by fusion [27]. We do not recall here their construction and give only their explicit form

$$R^{(s, s')}(u) = \sum_{k=|s-s'|}^{s+s'} f_k^{(s, s')}(u) \mathcal{P}_k^{(s, s')} \quad (3.2)$$

with $f_k^{(s, s')}(u) = \prod_{\ell=k+1}^{s+s'} \left(\frac{u - i\ell}{u + i\ell} \right)$. As usual, one have introduced the following projectors

$$\mathcal{P}_k^{(s, s')} = \prod_{\substack{j=|s-s'| \\ j \neq k}}^{s+s'} \frac{(\pi_s \otimes \pi_{s'})(e_3 \otimes e_3 + \frac{1}{2}(e_+ \otimes e_- + e_- \otimes e_+)) - x_j}{x_k - x_j} \quad (3.3)$$

with $x_k = \frac{1}{2}[k(k+1) - s(s+1) - s'(s'+1)]$. In particular, for $s = s' = \frac{1}{2}$, we get the usual Yang's R-matrix [7]

$$R_{12}(u) = R_{12}^{(\frac{1}{2}, \frac{1}{2})}(u) = \frac{1}{u+i}(u + iP_{12}). \quad (3.4)$$

The normalization has been chosen such that it leads to regular and unitary matrices:

$$R_{0,i}^{(s, s)}(0) = P_{0,i}^{(s)} \quad \text{and} \quad R_{0,j}^{(s_i, s_j)}(u) R_{0,j}^{(s_i, s_j)}(-u) = 1 \quad (3.5)$$

where $P_{0,i}^{(s)}$ is the permutation operator acting on $\mathbb{C}^{2s+1} \otimes \mathbb{C}^{2s+1}$. They satisfy also the famous Yang-Baxter equation [7, 8]:

$$R_{i,j}^{(s_i,s_j)}(u-v)R_{i,k}^{(s_i,s_k)}(u-w)R_{j,k}^{(s_j,s_k)}(v-w) = R_{j,k}^{(s_j,s_k)}(v-w)R_{i,k}^{(s_i,s_k)}(u-w)R_{i,j}^{(s_i,s_j)}(u-v). \quad (3.6)$$

Finally, we introduce the following transfer matrices

$$t^{(\bar{s}_i)}(u) = \text{tr}_0 T_0^{(\bar{s}_i)}(u). \quad (3.7)$$

They lead to conserved quantities

$$I_n^{(s)} = \frac{d^n}{du^n} \ln t^{(s)}(u) \Big|_{u=0} \quad \text{with } s \in \mathcal{S}. \quad (3.8)$$

The Hamiltonian is usually chosen as any linear combinations of the following conserved charges

$$H^{(s)} = iI_1^{(s)}, \quad (3.9)$$

where the factor i allows us to obtain a Hermitian operator. However, the explicit computation of this Hamiltonian for an L_0 -regular spin chain become more involved since $R^{(s,s')}(0)$ is not a permutation for $s \neq s'$. Even locality of this operator is not obvious. Fortunately, by introducing new transfer matrix, we may construct new framework where the usual constructions work even for a general spin chain. We illustrate that in the following subsection by computing the momentum and the Hamiltonian.

3.2 Momentum and Hamiltonian

In the homogeneous case ($L_0 = 1$), the transfer matrix at vanishing spectral parameter provides the one-step shift operator and the momentum is given by its logarithm. In the case of L_0 -regular spin chain, the one-step shift operator is not any more conserved. However, it is obvious that the L_0 -shift operator, \mathcal{S}_{L_0} , must be conserved. To express this operator in terms of the transfer matrices (3.7), we introduce the following transfer matrix

$$\mathbf{t}(\mathbf{u}) = t^{(s_1)}(u_1) t^{(s_2)}(u_2) \dots t^{(s_{L_0})}(u_{L_0}) \quad (3.10)$$

where u_1, \dots, u_{L_0} are different spectral parameters. Obviously, it commutes with any other transfer matrix $t^{(s_i)}(v)$ and it may be written as follows

$$\mathbf{t}(\mathbf{u}) = \text{tr}_{a_1, \dots, a_{L_0}} \prod_{0 \leq p < L/L_0}^{\rightarrow} \mathcal{R}_{(a_1, \dots, a_{L_0}), (1+pL_0, \dots, (p+1)L_0)}(\mathbf{u}) \quad (3.11)$$

where we have introduced

$$\mathcal{R}_{(a_1, \dots, a_{L_0}), (b_1, \dots, b_{L_0})}(\mathbf{u}) = \left(R_{a_1, b_1}^{(s_1, s_1)}(u_1) \dots R_{a_{L_0}, b_1}^{(s_{L_0}, s_1)}(u_{L_0}) \right) \dots \left(R_{a_1, b_{L_0}}^{(s_1, s_{L_0})}(u_1) \dots R_{a_{L_0}, b_{L_0}}^{(s_{L_0}, s_{L_0})}(u_{L_0}) \right). \quad (3.12)$$

The importance of this new operator lies in the fact that it is regular i.e.

$$\mathcal{R}_{(a_1, \dots, a_{L_0}), (b_1, \dots, b_{L_0})}(\mathbf{u}) \Big|_{\mathbf{u}=\mathbf{0}} = P_{a_1, b_1}^{(s_1)} P_{a_2, b_2}^{(s_2)} \dots P_{a_{L_0}, b_{L_0}}^{(s_{L_0})} . \quad (3.13)$$

To prove this regularity, we have used the regularity of the R-matrix as well as the unitarity relation for the vanishing spectral parameter (see relations (3.5)). Using this property, it is a standard computation to show that $\mathbf{t}(\mathbf{0})$ provides the L_0 -step shift operator \mathcal{S}_{L_0} . We can deduce from this operator, the momentum operator $\widehat{\mathbf{p}}$ defined as

$$\mathbf{t}(\mathbf{u}) \Big|_{\mathbf{u}=\mathbf{0}} = \mathcal{S}_{L_0} = \exp(-iL_0 \widehat{\mathbf{p}}) . \quad (3.14)$$

It is easy to shown that the gradient of the transfer matrix $\mathbf{t}(\mathbf{u})$ allows us to obtain the general Hamiltonian. Indeed, we get

$$H = i\boldsymbol{\alpha} \cdot \nabla \ln \mathbf{t}(\mathbf{u}) \Big|_{\mathbf{u}=\mathbf{0}} \quad \text{with} \quad \boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{L_0}) \quad \text{and} \quad \nabla = \begin{pmatrix} \frac{\partial}{\partial u_1} \\ \vdots \\ \frac{\partial}{\partial u_{L_0}} \end{pmatrix} \quad (3.15)$$

where $\alpha_1, \dots, \alpha_{L_0}$ are free parameters. Using definition (3.10) of the transfer matrix $\mathbf{t}(\mathbf{u})$, we can show that H is a linear combination of $H^{(s)}$

$$H = \sum_{s \in \mathcal{S}} \theta_s H^{(s)} , \quad (3.16)$$

where $\theta_s = \sum_{j=1}^{L_0} \delta_{s, s_j} \alpha_j$. In the following, we will take $\theta_s > 0$, in order to have a correct particle interpretation for our models². Locality of this general Hamiltonian can be seen using the following explicit formula

$$H = i \sum_{p=1}^{L/L_0} P_{1+(p-1)L_0, 1+pL_0} \dots P_{pL_0, (p+1)L_0} \boldsymbol{\alpha} \cdot \nabla \mathcal{R}_{(1+(p-1)L_0, \dots, pL_0), (1+pL_0, \dots, (p+1)L_0)}(\mathbf{u}) \Big|_{\mathbf{u}=\mathbf{0}} . \quad (3.17)$$

4 Bethe ansatz

4.1 Bethe equations

To obtain the spectrum of the Hamiltonians H , we study, as usual, the spectrum of the transfer matrix $t^{(s)}(u)$. Its eigenvalues $\tau^{(s)}(\lambda)$ have been computed by algebraic Bethe ansatz in [30, 31]

$$\tau^{(s)}(u) = \sum_{\alpha=0}^{2s} C_{\alpha}^{(s)}(u) \prod_{p=1}^M \frac{(u - \lambda_p + i(s+1))(u - \lambda_p - is)}{(u - \lambda_p + i(\alpha - s + 1))(u - \lambda_p + i(\alpha - s))} \quad (4.1)$$

²The ground state configuration is not unique when one of the coefficients θ_s vanishes, see [28] where the particular case of alternating spin chain is studied. The case of negative coefficients have been studied in [29], still for an alternating spin chain.

where

$$C_\alpha^{(s)}(u) = \prod_{k=\alpha}^{2s-1} \prod_{s' \in \mathcal{S}} \left(\frac{u + i(k - s - s' + 1)}{u + i(k - s + s' + 1)} \right)^{L\rho_{s'}}, \quad \alpha < 2s \quad \text{and} \quad C_{2s}^{(s)}(u) = 1. \quad (4.2)$$

The parameters $\{\lambda_n\}$ are the Bethe roots satisfying the Bethe equations:

$$\prod_{s \in \mathcal{S}} \left(\frac{\lambda_n + is}{\lambda_n - is} \right)^{L\rho_s} = - \prod_{p=1}^M \frac{\lambda_n - \lambda_p + i}{\lambda_n - \lambda_p - i} \quad \text{for } 1 \leq n \leq M. \quad (4.3)$$

and M is an integer depending on the choice of the eigenvectors. These parameters are linked to the total spin of the chain [18]³

$$S = S_0 - M. \quad (4.4)$$

where $S_0 = L \sum_{s \in \mathcal{S}} s \rho_s$ is the highest spin reached in this model. Let us remark that the Bethe equations do not depend on the choice of the Hamiltonian.

The momentum \mathbf{p} (eigenvalues of $\hat{\mathbf{p}}$) and the energies, $E^{(s)}$ (eigenvalues of the Hamiltonian $H^{(s)}$), are given by

$$\mathbf{p} = i \sum_{s \in \mathcal{S}} \rho_s \sum_{n=1}^M \ln \left(\frac{\lambda_n + is}{\lambda_n - is} \right) \bmod \left(\frac{2\pi}{L_0} \right) = \sum_{s \in \mathcal{S}} \rho_s \sum_{n=1}^M (\varphi_{2s}(\lambda_n) + \pi) \bmod \left(\frac{2\pi}{L_0} \right) \quad (4.5)$$

$$E^{(s)} = - \sum_{k=1}^M \frac{2s}{(\lambda_k)^2 + s^2} \quad (4.6)$$

Let us remark that each λ_k provides a negative energy. Then, the state with $M = 0$, which is the pseudo-vacuum used in the procedure of the algebraic Bethe ansatz, is the state with highest energy. We are in the case of an 'anti-ferromagnetic' spin chain. The true vacuum will be studied in the following. Multiplying the Hamiltonian by a negative constant, we describe a 'ferromagnetic' spin chain.

4.2 String hypothesis

We want to study the previous models in the thermodynamical limit ($L \rightarrow \infty$) and, in particular, to compute the energy of the vacuum state as well as the one of the first excited states. In the thermodynamical limit, it is usual to use the string hypothesis which states that all the Bethe roots $\{\lambda_p, p = 1, \dots, M\}$ gather into ν_m strings of length $2m$, called $2m$ -strings, ($m \in \frac{1}{2}\mathbb{Z}_{>0}$) of the following form

$$\lambda_{m,k} + i\alpha, \quad \alpha = -m + \frac{1}{2}, -m + \frac{3}{2}, \dots, m - \frac{1}{2} \quad (4.7)$$

where $k = 1, \dots, \nu_m$ and $\lambda_{m,k}$, the center of the string, is real. We get

$$M = 2 \sum_{m \in \frac{1}{2}\mathbb{Z}_{>0}} m \nu_m \quad \text{and} \quad S = -2 \sum_{m \in \frac{1}{2}\mathbb{Z}_{>0}} m \nu_m + L \sum_{s \in \mathcal{S}} \rho_s s. \quad (4.8)$$

³Be careful, there is a factor 2 between the spin S defined here and the one defined in [18].

Remark 4.1 *In the string hypothesis, usually, we suppose also that the finite size effects in the imaginary part are exponentially small in L . However, it is well-established that this assumption is wrong for spin s chains with $s \geq 1$ (see, for example, the articles [32, 33] where this deviation has been computed numerically and analytically). The general case treated here is certainly worst and, in general, the decay of the imaginary part will be of order $1/L$. However, to study the Bethe equations in this hypothesis is still interesting and fruitful. Indeed, the number of states obtained by this way is in agreement with the dimension of the Hilbert space (see section 4.3), we can compute the energy of the antiferromagnetic vacuum and determine the dispersion relation for the first excited states.*

Within this hypothesis, the Bethe equations (4.3) can be transformed and become equations in terms of the real centers of the strings only. After taking the logarithm, we get, for $m \in \frac{1}{2}\mathbb{Z}_{>0}$ and $k = 1, \dots, \nu_m$,

$$-2\pi Q_{m,k} + L \sum_{s \in \mathcal{S}} \rho_s \Phi_{2s}^{(m)}(\lambda_{m,k}) = \sum_{p \in \frac{1}{2}\mathbb{Z}_{>0}} \sum_{\ell=1}^{\nu_p} \Phi_2^{(p,m)}(\lambda_{m,k} - \lambda_{p,\ell}) \quad (4.9)$$

where $Q_{m,k}$ are half-integers and

$$\Phi_2^{(p,m)}(\lambda) = \varphi_{2p+2m}(\lambda) + \varphi_{2|p-m|}(\lambda) + 2 \sum_{\alpha=|p-m|+1}^{p+m-1} \varphi_{2\alpha}(\lambda) \quad (4.10)$$

$$\Phi_p^{(m)}(\lambda) = \sum_{\alpha=|\frac{p}{2}-m+\frac{1}{2}|+1}^{\frac{p}{2}+m-\frac{1}{2}} \varphi_{2\alpha}(\lambda) + \theta(p > 2m-1) \varphi_{p-2m+1}(\lambda), \quad p \in \mathbb{Z}_{>0}, \quad m \in \frac{1}{2}\mathbb{Z}_{>0}. \quad (4.11)$$

The numbers $Q_{m,k}$ are supposed to be quantum numbers i.e. for one set there exists one and only one solution to Bethe equations (4.9). Constraints on these numbers will be given in section 4.3.

Within the string hypothesis, the momentum and the energies (4.6) become, for $s \in \mathcal{S}$,

$$\mathfrak{p} = \sum_{s \in \mathcal{S}} \rho_s \sum_{m \in \frac{1}{2}\mathbb{Z}_{>0}} \sum_{k=1}^{\nu_m} \Phi_{2s}^{(m)}(\lambda_{m,k}) + 2\pi \sum_{m \in \frac{1}{2}\mathbb{Z}_{>0}} m \nu_m \mod\left(\frac{2\pi}{L_0}\right), \quad (4.12)$$

$$E^{(s)} = - \sum_{m \in \frac{1}{2}\mathbb{Z}_{>0}} \sum_{k=1}^{\nu_m} \sum_{\alpha=-m+\frac{1}{2}}^{m-\frac{1}{2}} \frac{2(\alpha+s)}{(\lambda_{m,k})^2 + (\alpha+s)^2} = - \sum_{m \in \frac{1}{2}\mathbb{Z}_{>0}} \sum_{k=1}^{\nu_m} \Psi_{2s}^{(m)}(\lambda_{m,k}) \quad (4.13)$$

where $\Psi_p^{(m)}(\lambda)$ is the derivative of $\Phi_p^{(m)}(\lambda)$. We will need also to define $\Psi_2^{(p,m)}(\lambda)$, the derivative of $\Phi_2^{(p,m)}(\lambda)$. The explicit form of these functions may be found in [18].

4.3 Valence and completeness of Bethe states

From equation (4.9) we can get bounds on $Q_{m,k}$ [18]:

$$Q_{m,\max} = \frac{1}{2} \left(\nu_m - 1 + 2 \sum_{i=1}^L \min(m, s_i) - 4 \sum_{n \in \frac{1}{2}\mathbb{Z}^+} \min(m, n) \nu_n \right) \quad \text{and} \quad Q_{m,\min} = -Q_{m,\max}. \quad (4.14)$$

Now we can define the valence, which is the number of allowed quantum numbers Q_m for a given configuration $\{\nu\}$:

$$P_m(\nu) = 2Q_{m,max} + 1 = 2 \sum_{j=1}^L \min(m, s_j) - 4 \sum_{n \in \frac{1}{2}\mathbb{Z}^+} \min(m, n) \nu_n + \nu_m \quad (4.15)$$

As explained previously, to each set of quantum numbers Q corresponds one Bethe eigenstate. Then, to be sure that this method gives all the eigenstates, we must prove that the number of eigenstates obtained by Bethe ansatz is equal to the dimension of the starting Hilbert space. Let us recall that the Bethe eigenvectors are highest weight for the $gl(2)$ symmetry, so that a Bethe eigenvector is $(2S+1)$ -degenerated, where $S = S_0 - M$ is its total spin (4.4). Thus, given the valence (4.15), the number of the eigenstates for a given M obtained by Bethe ansatz is

$$Z_M^{bethe} = (2S_0 - 2M + 1) \sum_{\substack{\{\nu_m\} \\ 2 \sum k \nu_k = M}} \prod_{m \in \frac{1}{2}\mathbb{Z}^+} \binom{P_m(\nu)}{\nu_m} \quad (4.16)$$

where we sum over all the possible configurations $\{\nu\}$ (number of string of each type) and $\binom{a}{c}$ is the binomial coefficient.

Following the previous work [34] on this problem, we can compute explicitly this number. The proof is based on the following combinatorial identity, for $\{b\}$ a set of real numbers and $\{\nu\}$ a set of positive integers,

$$\sum_{M=0}^{\infty} Z(\{b\}, M) x^M = (1-x) \prod_{n=1}^{\infty} (1-x^n)^{b_n} \quad (4.17)$$

where we have introduced

$$Z(\{b\}, M) = \sum_{\substack{\{\nu_m\} \\ 2 \sum_k k \nu_k = M}} \prod_{m \in \frac{1}{2}\mathbb{Z}^+} \binom{A_m(\nu, b)}{\nu_m} \quad (4.18)$$

$$A_m(\nu, b) = - \sum_{j=1}^{2m} (2m-j+1) b_j - 2M + 4 \sum_{n>m} (n-m) \nu_n + \nu_m \quad (4.19)$$

For the following particular choice of the set $\{b\}$

$$b_1 = -L \quad (4.20)$$

$$b_m = \begin{cases} 0 & , \quad m \neq 2\bar{s}_j + 1 \\ L\rho_j & , \quad m = 2\bar{s}_j + 1 \end{cases} \quad m = 2, 3, \dots \quad (4.21)$$

we have $A_m(\nu, b) = P_m(\nu)$ and (4.17) in the limit $x \rightarrow 1$ gives (see [34] for details):

$$\sum_{M=0}^{S_0} Z_M^{bethe} = \prod_{j=1}^{\mathcal{L}} (2\bar{s}_j + 1)^{L\rho_j} \quad (4.22)$$

The L.H.S. is the total number of states we get from the Bethe equations in the string hypothesis, while the R.H.S. is the total dimension of the Hilbert space. Thus, the Bethe ansatz in the string hypothesis leads to a complete basis of states.

5 Vacuum state

For any choice of Hamiltonian $H^{(s)}$ (or any linear combination with positive coefficients), the contribution to the energy of any Bethe roots is negative (see eq. (4.6) or (4.13)). Then, to obtain the true ground state (i.e. to minimize the energy), we look for a configuration with a maximum number of roots. So, it is natural to introduce the vacuum state defined by

$$P_n(\nu) - \nu_n = 0 \quad \text{for } n \in \frac{1}{2}\mathbb{Z}_{>0}. \quad (5.1)$$

where the valences $P_n(\nu)$ have been defined by (4.15). This constraint has been solved in [18] and one finds a unique configuration characterized by

$$\nu_s = \begin{cases} \frac{L\rho_s}{2} & s \in \mathcal{S} \\ 0 & \text{otherwise} \end{cases} \quad (5.2)$$

One interprets it as \mathcal{L} filled Fermi seas of $2s$ -string (for $s \in \mathcal{S}$). From now on, this state becomes the reference state. As we will see in section 6, any excited states have an energy greater than the one of this reference state, whatever the Hamiltonian $H^{(s)}$ one considers: the vacuum state is also its ground state. It is non degenerate since its spin vanishes (which is easily deduced from (4.8)).

Relation (5.1) implies that the quantum numbers fulfil all the possibilities:

$$Q_{s,k} = k - \frac{1}{2} - \frac{1}{2}\nu_s \quad \text{with } k = 1, \dots, \nu_s \quad \text{and } s \in \mathcal{S}. \quad (5.3)$$

For the vacuum state, in the thermodynamical limit, the Bethe roots $\{\lambda_{s,k} \mid k = 1, \dots, \nu_s, s \in \mathcal{S}\}$ become dense in \mathbb{R} and we can replace them by their density $\sigma_s^{(0)}(\lambda)$. Then, the Bethe equations (4.9) can be transformed to the following integral equations, for $s \in \mathcal{S}$,

$$-2\pi\sigma_s^{(0)}(\lambda_0) + \sum_{r \in \mathcal{S}} \rho_r \Psi_{2r}^{(s)}(\lambda_0) = \sum_{r \in \mathcal{S}} \int_{-\infty}^{\infty} d\lambda \sigma_r^{(0)}(\lambda) \Psi_2^{(r,s)}(\lambda_0 - \lambda). \quad (5.4)$$

Solving these integral equations, we get the densities [18]

$$\sigma_s^{(0)}(\lambda) = \frac{\rho_s}{2} \frac{1}{\cosh(\pi\lambda)} = \rho_s \sigma^{(0)}(\lambda). \quad (5.5)$$

The computation of these densities allows us to determine the energies of the vacuum

Theorem 5.1 *The energies per site (energy densities), eigenvalues of the Hamiltonians $H^{(s)}$ ($s \in \mathcal{S}$) divided by the length L , for an L_0 -regular spin chain are given by*

$$\mathcal{E}_0^{(s)} = - \sum_{s' \in \mathcal{S}} \rho_{s'} \left(\psi\left(\frac{s' + s + 1}{2}\right) - \psi\left(\frac{|s' - s| + 1}{2}\right) \right) \quad (5.6)$$

where $\psi(x)$ is the Euler digamma functions.

Proof: Replacing the sum $\sum_{k=1}^{\nu_m}$ in (4.13) by an integral, we get the energies for the vacuum

$$E_0^{(s)} = -L \sum_{s' \in \mathcal{S}} \rho_{s'} \int_{-\infty}^{\infty} d\lambda \sigma^{(0)}(\lambda) \Psi_{2s}^{(s')}(\lambda) = -2\pi L \sum_{s' \in \mathcal{S}} \rho_{s'} \int_{-\infty}^{\infty} dp \hat{\sigma}^{(0)}(p) \hat{\Psi}_{2s}^{(s')}(p). \quad (5.7)$$

The second equality is obtained via the Plancherel's theorem. Using the explicit forms of $\hat{\Psi}$ [18] and the one of $\hat{\sigma}^{(0)}$ (see 5.5), we get the result. \blacksquare

Similarly, we can prove that, for the vacuum, the momentum is given by

$$\mathfrak{p}_0 = \pi L \sum_{s \in \mathcal{S}} s \rho_s \bmod\left(\frac{2\pi}{L_0}\right). \quad (5.8)$$

6 Excited states

6.1 Characterization of excited states

The excited states are obtained by creating holes in the filled Fermi seas of $2s$ -strings ($s \in \mathcal{S}$) or creating new $2r$ -strings with $r \in \mathcal{R}$. Such states are characterized by the following configuration:

$$\tilde{\nu}_s = \nu_s - \mu_s \quad \text{for } s \in \mathcal{S} \quad \text{and} \quad \tilde{\nu}_r \geq 0 \quad \text{for } r \in \mathcal{R}, \quad (6.1)$$

where we kept the notation ν_s ($s \in \mathcal{S}$) for the vacuum configuration (5.2) while the positive integers $\tilde{\nu}_r$ ($r \in \mathcal{R}$) correspond to the numbers of new $2r$ -strings with centers $\lambda_{r,\ell}$ ($\ell = 1, \dots, \tilde{\nu}_r$). The corresponding valences are given by, for $n \in \frac{1}{2}\mathbb{Z}_{>0}$,

$$\tilde{P}_n(\tilde{\nu}) = \tilde{\nu}_n + 4 \sum_{s \in \mathcal{S}} \min(n, s) \mu_s - 4 \sum_{r \in \mathcal{R}} \min(n, r) \tilde{\nu}_r. \quad (6.2)$$

Since \tilde{P}_s depends on $\tilde{\nu}$, μ_s is not the number of holes in the sea of $2s$ -strings: this physical quantity is rather defined by

$$\mathcal{D}_s = \tilde{P}_s(\tilde{\nu}) - \tilde{\nu}_s \quad \text{for } s \in \mathcal{S}. \quad (6.3)$$

It is the number of unused values, $\tilde{Q}_{s,d}$ ($d = 1, \dots, \mathcal{D}_s$), in the set $\{\frac{1-\tilde{P}_s(\tilde{\nu})}{2}, \dots, \frac{\tilde{P}_s(\tilde{\nu})-1}{2}\}$ of possible choices for the quantum numbers in the sea of $2s$ -string. We denote by $\mathcal{D} = \sum_{s \in \mathcal{S}} \mathcal{D}_s$ the total number of holes. In the same way that one associates a unique Bethe root, $\lambda_{s,k}$, to each quantum number $Q_{s,k}$, we introduce $\tilde{\lambda}_{s,d}$ associated to $\tilde{Q}_{s,d}$. These numbers $\tilde{\lambda}_{s,d}$ can be interpreted as rapidities of holes. Let us remark that \mathcal{D}_s ($s \in \mathcal{S}$) is always even (see equation (6.2)). This means that a single excitation is composed of two holes. This behavior appears already in the usual homogeneous spin $\frac{1}{2}$ spin chain [22]. To simplify some formulas, we will use also shorter notations $\mathcal{D}_j = \mathcal{D}_{\bar{s}_j}$, $\mu_j = \mu_{\bar{s}_j}$ and $\tilde{\lambda}_{j,d} = \tilde{\lambda}_{\bar{s}_j,d}$.

Let us remark that the numbers $\{\mu\}$ are determined by the \mathcal{D} 's and $\tilde{\nu}_r$'s which allows us to express the numbers of unused quantum numbers $\mathcal{A}_r = \tilde{P}_r(\tilde{\nu}) - \tilde{\nu}_r$ for the new strings by

$$\mathcal{A}_r = \frac{r - \bar{s}_j}{\bar{s}_{j+1} - \bar{s}_j} \mathcal{D}_{j+1} + \frac{\bar{s}_{j+1} - r}{\bar{s}_{j+1} - \bar{s}_j} \mathcal{D}_j - 4 \sum_{m \in \mathcal{R}_j} \frac{(\bar{s}_{j+1} - \max(m, r))(\min(m, r) - \bar{s}_j)}{\bar{s}_{j+1} - \bar{s}_j} \tilde{\nu}_m, \quad r \in \mathcal{R}_j \quad (6.4)$$

Let us remark that the numbers \mathcal{A}_r is always even (see relation (6.2)).

Now, the number of eigenstates for a given number of holes is given by

$$Z(\{\mathcal{D}\}) = \sum_{\{\tilde{\nu}_r\}} (2S+1) \prod_{r \in \mathcal{R}} \binom{\mathcal{A}_r + \tilde{\nu}_r}{\tilde{\nu}_r} \quad (6.5)$$

where we sum over all the sets $\{\tilde{\nu}_r \in \mathbb{Z}_{\geq 0} | r \in \mathcal{R}\}$ such that

$$\mathcal{A}_r = \tilde{P}_r(\tilde{\nu}) - \tilde{\nu}_r \geq 0 \quad \text{for } r \in \mathcal{R}. \quad (6.6)$$

Inequality (6.6) translates the obvious fact the we cannot have more quantum numbers than allowed. The factor $2S+1$ comes from the degeneracy due to the $gl(2)$ symmetry with the total spin rewritten as follows

$$S = \frac{\mathcal{D}_{\mathcal{L}}}{2} - 2 \sum_{r \in \mathcal{R}_{\mathcal{L}}} (r - \bar{s}_{\mathcal{L}}) \tilde{\nu}_r. \quad (6.7)$$

Then, to simplify relation (6.5), we invert relation (6.4) to get, for $r \in \mathcal{R}_j$

$$\tilde{\nu}_r = \frac{1}{2}(\mathcal{A}_{r-\frac{1}{2}} + \mathcal{A}_{r+\frac{1}{2}} - 2\mathcal{A}_r) \quad (6.8)$$

with the conventions $\mathcal{A}_s = \mathcal{D}_s$ (for $s \in \mathcal{S}$) and $\mathcal{A}_0 = 0$. Therefore,

$$Z(\{\mathcal{D}\}) = \prod_{j=0}^{\mathcal{L}} Z_j \quad \text{with} \quad Z_j = \sum_{\mathcal{A}_{\bar{s}_j+\frac{1}{2}}, \dots, \mathcal{A}_{\bar{s}_{j+1}-\frac{1}{2}} \in 2\mathbb{Z}_{\geq 0}} \prod_{r \in \mathcal{R}_j} \binom{\frac{1}{2}(\mathcal{A}_{r-\frac{1}{2}} + \mathcal{A}_{r+\frac{1}{2}})}{\mathcal{A}_r}. \quad (6.9)$$

Finally, one can conjecture that these numbers are equal to, for $0 \leq j \leq \mathcal{L}-1$,

$$Z_j = \frac{2^{\mathcal{D}_j + \mathcal{D}_{j+1}}}{\bar{s}_{j+1} - \bar{s}_j + 1} \sum_{q=1}^{2\bar{s}_{j+1} - 2\bar{s}_j + 1} \sin^2 \left(\frac{q\pi}{2\bar{s}_{j+1} - 2\bar{s}_j + 2} \right) \cos^{\mathcal{D}_j + \mathcal{D}_{j+1}} \left(\frac{q\pi}{2\bar{s}_{j+1} - 2\bar{s}_j + 2} \right). \quad (6.10)$$

We do not know a full analytical proof of this result, but we proved it for $\bar{s}_{j+1} - \bar{s}_j = 1, \frac{3}{2}, 2, \dots, \frac{7}{2}$ by brute force calculations on binomial coefficients that we do not wish to reproduce here. Remark that a similar feature appears also in the counting of states for the homogeneous highest spin XXZ model studied in [35] (see also section 7.3). Finally, we can also obtain an exact closed form for $Z_{\mathcal{L}}$ given by

$$Z_{\mathcal{L}} = 2^{\mathcal{D}_{\mathcal{L}}}. \quad (6.11)$$

6.2 Density of roots for excited states

Now, we are in position to compute the densities of the Bethe roots corresponding to the states defined in the previous subsection. For the configuration (6.1), the Bethe equations (4.9) for $m \in \mathcal{S}$, in the thermodynamical limit, provides a linear integral equation for the densities $\sigma_s(\lambda)$ of $2s$ -strings:

$$\begin{aligned} -2\pi \left[\sigma_s(\lambda_0) + \frac{1}{L} \sum_{d=1}^{\mathcal{D}_s} \delta(\lambda_0 - \tilde{\lambda}_{s,d}) \right] + \sum_{s' \in \mathcal{S}} \rho_{s'} \int_{-\infty}^{\infty} \Psi_{2s'}^{(s)}(\lambda) \sigma_{s'}(\lambda) d\lambda \\ = \sum_{s' \in \mathcal{S}} \int_{-\infty}^{\infty} \Psi_2^{(s',s)}(\lambda_0 - \lambda) \sigma_s(\lambda) d\lambda + \frac{1}{L} \sum_{r \in \mathcal{R}} \sum_{\ell=1}^{\tilde{\nu}_r} \Psi_2^{(r,s)}(\lambda_0 - \lambda_{r,\ell}) \end{aligned} \quad (6.12)$$

There are also other Bethe equations for $\lambda_{r,\ell}$, $r \in \mathcal{R}$. We postpone their study in section 7.2.

These densities can be computed [21]:

$$\sigma_s(\lambda) = \sigma_s^{(0)}(\lambda) + \frac{1}{L}(\mathbf{r}_s(\lambda) + \mathbf{c}_s(\lambda)) \quad (6.13)$$

where $\sigma_s^{(0)}(\lambda)$ is the density (5.5) of the vacuum, $\mathbf{r}_s(\lambda)$ is the correction due to the holes and $\mathbf{c}_s(\lambda)$ is the polarization due to the new strings. The explicit form of these corrections [21] reads:

$$\begin{aligned} \mathbf{r}_{\bar{s}_j}(\lambda) = & \frac{1}{2\pi} \left(\sum_{d=1}^{\mathcal{D}_{j-1}} \kappa_{2(\bar{s}_j - \bar{s}_{j-1})-1}^{(\hbar_{j-1})}(\lambda - \tilde{\lambda}_{j-1,d}) + \sum_{d=1}^{\mathcal{D}_{j+1}} \kappa_{2(\bar{s}_{j+1} - \bar{s}_j)-1}^{(\hbar_j)}(\lambda - \tilde{\lambda}_{j+1,d}) \right. \\ & \left. + \sum_{d=1}^{\mathcal{D}_j} \left(\kappa_1^{(\hbar_j)}(\lambda - \tilde{\lambda}_{j,d}) + \kappa_1^{(\hbar_{j-1})}(\lambda - \tilde{\lambda}_{j,d}) - 2\pi\delta(\lambda - \tilde{\lambda}_{j,d}) \right) \right) \end{aligned} \quad (6.14)$$

$$\mathbf{c}_{\bar{s}_j}(\lambda) = -\frac{1}{2\pi} \sum_{m \in \mathcal{R}_{j-1}} \sum_{\ell=1}^{\tilde{\nu}_m} \gamma_{2(\bar{s}_j - m)}^{(\hbar_{j-1})}(\lambda - \lambda_{m,\ell}) - \frac{1}{2\pi} \sum_{m \in \mathcal{R}_j} \sum_{\ell=1}^{\tilde{\nu}_m} \gamma_{2(m - \bar{s}_j)}^{(\hbar_j)}(\lambda - \lambda_{m,\ell}) \quad (6.15)$$

where we have introduced $\hbar_j = \frac{\pi}{2(\bar{s}_{j+1} - \bar{s}_j)}$. We set by convention $\mathcal{D}_0 = 0 = \mathcal{D}_\infty$ (we recall the conventions $\bar{s}_0 = 0$ and $\bar{s}_{\mathcal{L}+1} = \infty$).

6.3 Energy and dispersion law

The densities given in previous section 6.2 allow us to compute the contribution at order $1/L$ of the first excited states to the energies.

Theorem 6.1 *The energy densities at order $1/L$ for the configuration (6.1) are $\mathcal{E}^{(s)} = \mathcal{E}_0^{(s)} + \frac{1}{L}\Delta E^{(s)}$ (for $s \in \mathcal{S}$) with $\mathcal{E}_0^{(s)}$ given in theorem 5.1 and*

$$\Delta E^{(s)} = \sum_{d=1}^{\mathcal{D}_s} \frac{\pi}{\cosh(\pi \tilde{\lambda}_{s,d})}. \quad (6.16)$$

Proof: There are three contributions to the energies due to \mathbf{r} , \mathbf{c} and $\lambda_{m,\ell}$ (with $m \notin \mathcal{S}$ and $s \in \mathcal{S}$):

$$\Delta E^{(s)} = -2\pi \sum_{s' \in \mathcal{S}} \int_{-\infty}^{\infty} dp (\widehat{\mathbf{r}}_{s'}(p) + \widehat{\mathbf{c}}_{s'}(p)) \widehat{\Psi}_{2s}^{(s')}(p) - \sum_{r \in \mathcal{R}} \sum_{k=1}^{\tilde{\nu}_r} \Psi_{2s}^{(r)}(\lambda_{r,k}). \quad (6.17)$$

Using the explicit forms (6.14) and (6.15) of $\widehat{\mathbf{r}}$ and \mathbf{c} , we prove the theorem. ■

Let us emphasize the remarkable simplicity of this result although we deal with any L_0 -regular $gl(2)$ spin chain. We remark that the contribution to the energy $E^{(\bar{s}_j)}$ (eigenvalues of the Hamiltonian constructed from the monodromy matrix with the auxiliary space in the spin \bar{s}_j) involves only the holes in the sea of strings of length $2\bar{s}_j$. The holes in the other seas as well as the new strings have a vanishing energy.

Similarly, we can compute the eigenvalues of the impulsion and we obtain

$$\mathbf{p} = \mathbf{p}_0 + \sum_{s \in \mathcal{S}} \rho_s \sum_{d=1}^{\mathcal{D}_s} \left(\arctan(\sinh(\pi \tilde{\lambda}_{s,d})) + \frac{\pi}{2} \right) = \mathbf{p}_0 + \sum_{s \in \mathcal{S}} \sum_{d=1}^{\mathcal{D}_s} \mathbf{p}^{(s)}(\tilde{\lambda}_{s,d}), \quad (6.18)$$

where we have introduced

$$\mathbf{p}^{(s)}(\lambda) = \rho_s \arctan(\sinh(\pi \lambda)) + \frac{\rho_s \pi}{2}. \quad (6.19)$$

We recall that \mathbf{p}_0 is the momentum of the vacuum defined by (5.8). Let us remark that $0 < \mathbf{p}^{(s)} < \rho_s \pi$. Then, we can deduce the dispersion law for these excited states, for $s \in \mathcal{S}$,

$$\Delta E^{(s)} = \pi \sum_{d=1}^{\mathcal{D}_s} \sin \left(\frac{\mathbf{p}^{(s)}(\tilde{\lambda}_{s,d})}{\rho_s} \right). \quad (6.20)$$

Thus, we conclude that, for the Hamiltonian $H^{(s)}$, the speed of sound⁴ of the holes in the filled seas of $2s$ -string is equal to π/ρ_s whereas it is 0 for the holes in the seas of $2s'$ -string ($s' \in \mathcal{S}$ and $s' \neq s$).

Choosing $\theta_s = \rho_s$, we get the Hamiltonian with the energy $\Delta E = \sum_{s \in \mathcal{S}} \rho_s \Delta E^{(s)}$. In this case, all the holes have the same speed of sound ($= \pi$). Then, we deduce that this Hamiltonian is a good candidate to be described by a continuum model which is conformal (see e.g. [13, 25]).

6.4 Interpretation in terms of particles

For the general Hamiltonian (3.16), the excited states, characterized by \mathcal{D}_s holes in the seas of $2s$ -string, can be interpreted as \mathcal{D}_s particles like excitations with the dispersion law $E(p) = \pi \theta_s \sin(p/\rho_s)$. We will call them particle of type j when they correspond to a hole in the sea of $2\bar{s}_j$ -string ($j = 1, 2, \dots, \mathcal{L}$).

As explained at the end of section 6.1, for a given number of holes, the state is degenerated due to the different possibility for the numbers $\tilde{\nu}_r$ ($r \in \mathcal{R}$) of new strings. In terms of particles, this degeneracy is interpreted as a presence of an internal degree of freedom for the particles.

To understand degeneracy (6.11), we associate to each particle of type \mathcal{L} a spin $\frac{1}{2}$ under the $gl(2)$ symmetry algebra. This gives a space of dimension $2^{\mathcal{D}_{\mathcal{L}}}$ which is in agreement with (6.11).

The particles of type j ($j \neq \mathcal{L}$) are scalar under the $gl(2)$ symmetry algebra (see eq. (6.7): \mathcal{D}_j , $j \neq \mathcal{L}$, does not appear in the expression of the spin). However, to explain the degeneracy given by (6.10), we need to introduce new internal degrees of freedom for the particles. The same problem occurs already in the case of the homogeneous spin chain and has been solved in [24]. Generalizing this interpretation, we conjecture that the space with \mathcal{D}_1 particles of type 1, \mathcal{D}_2 particles of type 2, ..., $\mathcal{D}_{\mathcal{L}}$ particles of type \mathcal{L} is isomorphic to

$$\bigotimes_{q=1}^{\mathcal{L}} \left(\mathcal{H}^{RSOS}(\mathcal{D}_{q-1}; \mathcal{D}_q; \bar{s}_q - \bar{s}_{q-1}) \right) \otimes (\mathbb{C}^2)^{\otimes \mathcal{D}_{\mathcal{L}}} \quad (6.21)$$

⁴We remind that the speed of sound is defined as the derivative of the energy w.r.t. the momentum at the Fermi surface.

where $\mathcal{H}^{RSOS}(\mathcal{D}; \mathcal{D}'; \bar{s})$ is the space of the integer sequences $(a_0, a_1, \dots, a_{\mathcal{D}}; b_0, b_1, \dots, b_{\mathcal{D}'})$ with

$$0 \leq a_i \leq 2\bar{s} \quad ; \quad 0 \leq b_j \leq 2\bar{s} \quad (6.22)$$

$$\frac{a_{i+1} - a_i + 2\bar{s} - 1}{2} \in \{0, 1, \dots, 2\bar{s} - 1\} \quad ; \quad \frac{b_{j+1} - b_j + 1}{2} \in \{0, 1\} \quad (6.23)$$

$$2\bar{s} - 2 \leq a_j + a_{j+1} \leq 2\bar{s} + 2 \quad (6.24)$$

and the boundary conditions $a_0 = 0$, $a_{\mathcal{D}} = b_0$ and $b_{\mathcal{D}'} = 0$. In words, this space corresponds to a generalized RSOS model [36] with a restriction parameter given by $2\bar{s} + 2$ and for which the \mathcal{D} first sites have a jump of $2\bar{s} - 1$ whereas the \mathcal{D}' last sites have a jump of 1. We give 2 examples in Figures 2 and 3 of paths corresponding to integer sequences for the restriction parameter 2 as well as the number of such paths.

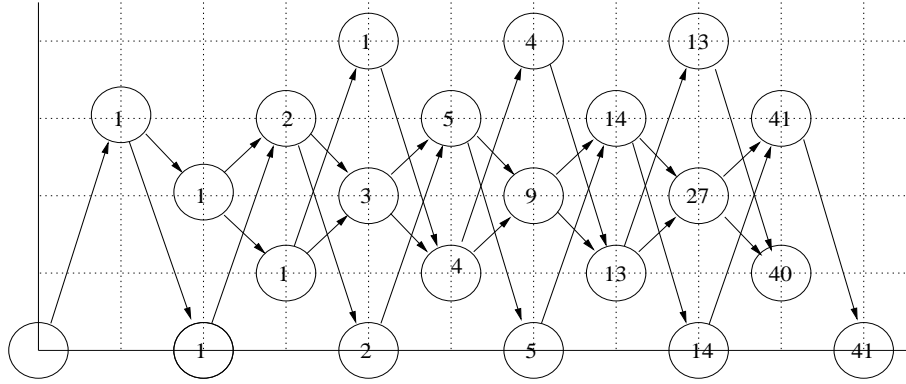


Figure 2: Path corresponding to $\mathcal{H}^{RSOS}(10; 0; 2)$. We have indicated the number of paths arriving to each allowed point.

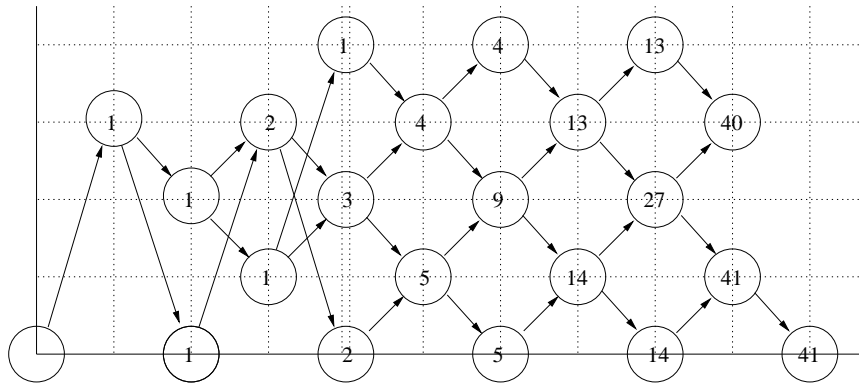


Figure 3: Path corresponding to $\mathcal{H}^{RSOS}(4; 6; 2)$. The double dashed line indicates the changes in the height of the jumps.

We show below that this interpretation is in agreement with the dimension of the spaces under consideration. We will show in section 7 that it is also compatible with the structure of the scattering matrix.

Proposition 6.2 *When \mathcal{D} and \mathcal{D}' are even, the number of states in $\mathcal{H}^{RSOS}(\mathcal{D}; \mathcal{D}'; \bar{s})$ is given by*

$$\frac{2^{\mathcal{D}+\mathcal{D}'}}{\bar{s}+1} \sum_{q=1}^{2\bar{s}+1} \sin^2\left(\frac{q\pi}{2\bar{s}+2}\right) \cos^{\mathcal{D}+\mathcal{D}'}\left(\frac{q\pi}{2\bar{s}+2}\right). \quad (6.25)$$

Proof: The computation for $\mathcal{H}^{RSOS}(0; \mathcal{D}'; \bar{s})$ has been done in [24] (see also [35]). Then, the proposition is demonstrated by remarking that the iterated relations satisfied by the number of paths at the even sites are identical for the jump 1 or jump $2\bar{s}-1$. ■

Let us stress that this number depends only on the sum $\mathcal{D} + \mathcal{D}'$. This point is illustrated by figures 2 and 3 where we can see that the numbers of paths at the even sites are the same in both figures.

The dimension of the spaces $\mathcal{H}^{RSOS}(\mathcal{D}; \mathcal{D}'; \bar{s})$ with $\mathcal{D} + \mathcal{D}' = \mathcal{D}_j + \mathcal{D}_{j+1}$ is in agreement with the degeneracy Z_j (see (6.10)). In the following, from the study of the S-matrix, we will show that the relevant models correspond to spaces $\mathcal{H}^{RSOS}(\mathcal{D}; \mathcal{D}'; \bar{s})$ with $\mathcal{D} = \mathcal{D}_j$, $\mathcal{D}' = \mathcal{D}_{j+1}$ and $\bar{s} = \bar{s}_{j+1} - \bar{s}_j$.

Finally, we introduce a basis for this space given by

$$E(\{a\}_{\mathcal{D}}; \{b\}_{\mathcal{D}'}) = E(a_0, a_1, \dots, a_{\mathcal{D}}; b_0, b_1, \dots, b_{\mathcal{D}'}) \quad (6.26)$$

where the sets of integers $\{a_i\}$ and $\{b_i\}$ satisfy constraints (6.22)-(6.24).

7 S-matrix

In this section, we want to compute the scattering matrix between the particles we introduced above. To do that, we follow the construction done in [37, 5]: one considers \mathcal{D} particles on a circle with a large (but finite) circumference L and computes the phase shift collected by one particle when it passes through all the other ones. Let us remark that, although the S-matrix is a bulk property, we must consider a finite system to compute it by this method.

We will get an expression (see equation (7.6)) which depends explicitly on the new Bethe roots $\lambda_{r,k}$ ($r \in \mathcal{R}$ and $k = 1, \dots, \tilde{\nu}_r$). These new Bethe roots are not free: they are determined by Bethe equations (4.9) for $m \in \mathcal{R}$ which were not used up to now (see section 7.2). Unfortunately, these equations cannot be solved in general. However, remarking that the equations we get are very similar to the ones of the homogeneous spin studied in [24], we will conjecture an explicit form for the scattering matrix.

7.1 \mathcal{D} -body scattering matrix

In general, for a given excited state with \mathcal{D} holes, the scattering matrix, $S_{s,d}$ ($s \in \mathcal{S}$ and $1 \leq d \leq \mathcal{D}_s$), between one particle with rapidity $\tilde{\lambda}_{s,d}$ and all the other particles is defined by the following finite volume quantization of momentum of this particle:

$$\exp(i\mathbf{p}^{(s)}(\tilde{\lambda}_{s,d})L) S_{s,d} = 1. \quad (7.1)$$

Taking the logarithm of the previous relation, we get

$$\mathfrak{p}^{(s)}(\tilde{\lambda}_{s,d}) + \frac{1}{L}\Phi_{s,d} = \frac{2\pi\tilde{Q}_{s,d}}{L} \quad (7.2)$$

where $S_{s,d} = \exp(i\Phi_{s,d})$ and $\tilde{Q}_{s,d}$ are the lacking quantum number corresponding to $\tilde{\lambda}_{s,d}$.

To find explicitly the form of Φ , let us remark that, using the definition of the density σ and its explicit form (6.13), we get

$$\frac{2\pi\tilde{Q}_{s,d}}{L} = 2\pi \int_{-\infty}^{\tilde{\lambda}_{s,d}} d\lambda \left[\sigma_s(\lambda) + \frac{1}{L} \sum_{q=1}^{\mathcal{D}_s} \delta(\lambda - \tilde{\lambda}_{s,q}) \right]. \quad (7.3)$$

Then, we can show, due to the explicit form of the density $\sigma^{(0)}$ (see (5.5)) and of the momentum (see (6.19)), that

$$\mathfrak{p}^{(s)}(\lambda) = 2\pi \int_{-\infty}^{\lambda} d\mu \sigma_s^{(0)}(\mu). \quad (7.4)$$

Finally, we deduce from (7.2), using (6.13) to express $\sigma_s(\lambda)$, that

$$\Phi_{s,d} = 2\pi \int_{-\infty}^{\tilde{\lambda}_{s,d}} d\lambda \left[\mathfrak{r}_s(\lambda) + \sum_{q=1}^{\mathcal{D}_s} \delta(\lambda - \tilde{\lambda}_{s,q}) + \mathfrak{c}_s(\lambda) \right]. \quad (7.5)$$

Now, we use the Fourier transform of the densities (6.14) and (6.15) to express the Fourier transform of the derivative (w.r.t. $\tilde{\lambda}_{s,d}$) of $\Phi_{s,d}$ in terms of the functions $\hat{\gamma}$ and $\hat{\kappa}$. Then, by using the results gather in section 2, we get the explicit form of the S-matrix of the d^{th} particle of type j up to a multiplicative constant C_j :

$$S_{j,d} = C_j \check{S}_{j,d}(\tilde{\lambda}_{\bar{s}_j,d}) \tilde{S}_{j,d}(\tilde{\lambda}_{\bar{s}_j,d}) \quad (7.6)$$

where

$$\check{S}_{j,d}(\lambda) = \prod_{m \in \mathcal{R}_{j-1}} \prod_{\ell=1}^{\tilde{\nu}_m} G_{2(\bar{s}_j-m)}^{(\hbar_{j-1})}(\lambda - \lambda_{m,\ell}) \prod_{m \in \mathcal{R}_j} \prod_{\ell=1}^{\tilde{\nu}_m} G_{2(m-\bar{s}_j)}^{(\hbar_j)}(\lambda - \lambda_{m,\ell}) \quad (7.7)$$

and

$$\tilde{S}_{j,d}(\lambda) = \prod_{q=1}^{\mathcal{D}_{j-1}} \mathcal{K}_{\frac{\pi}{\hbar_{j-1}}-1}^{(\hbar_{j-1})}(\tilde{\lambda}_{j-1,q} - \lambda) \prod_{q=1}^{\mathcal{D}_j} \mathcal{K}_1^{(\hbar_{j-1})}(\tilde{\lambda}_{j,q} - \lambda) \mathcal{K}_1^{(\hbar_j)}(\tilde{\lambda}_{j,q} - \lambda) \prod_{q=1}^{\mathcal{D}_{j+1}} \mathcal{K}_{\frac{\pi}{\hbar_j}-1}^{(\hbar_j)}(\tilde{\lambda}_{j+1,q} - \lambda) \quad (7.8)$$

The constant may be computed by

$$C_j = \lim_{\lambda \rightarrow -\infty} (\check{S}_{\bar{s}_j,d}(\lambda) \tilde{S}_{\bar{s}_j,d}(\lambda))^{-1}, \quad (7.9)$$

and, by using (2.11) and knowing that $\exp(i\pi\mathcal{D}_j) = 1$ (since \mathcal{D}_j is even), we get $C_j = \exp(-i\pi\mu_{\bar{s}_j})$. Let us recall that $\mu_{\bar{s}_j}$, defined in section 6.1, is integer. Then, the constant C_j is a sign.

We recall also that the Bethe roots $\lambda_{r,\ell}$ (for $r \in \mathcal{R}$) are functions of $\tilde{\lambda}_{\bar{s}_j,d}$ via the Bethe equations (7.10). Unfortunately, these equations are not solved explicitly in general, so that (7.7) for $\lambda = \tilde{\lambda}_{\bar{s}_j,d}$ cannot be brought explicitly to a factorized form depending on $\tilde{\lambda}_{\bar{s}_k,q} - \tilde{\lambda}_{\bar{s}_j,d}$ solely.

Remark 7.1 *The previous form of the scattering matrix, computed within the string hypothesis, proves that, for excitations with only holes and no new string, the scattering matrix factorizes (as expected since we study integrable systems). This factor is called usually the CDD factor of the scattering matrix. When we have also new strings ($\check{S} \neq 1$), the factorization should also occur but we could not prove it on the general explicit form (7.7). Indeed, one cannot (without solving (7.10)) write the scattering matrix as a product of functions of differences between hole rapidities. Nevertheless, the factorization of the scattering matrix is assumed since the system is integrable.*

7.2 Bethe equations between holes and new strings

There exist additional relations between the holes in the seas and the new strings provided by the Bethe equations (4.9) for $m \in \mathcal{R}$. They are given explicitly by, for $1 \leq j \leq \mathcal{L}$ and $m \in \mathcal{R}_j$,

$$-2\pi Q_{m,k} + \sum_{d=1}^{\mathcal{D}_j} \Gamma_{2(m-\bar{s}_j)}^{(\bar{h}_j)}(\lambda_{m,k} - \tilde{\lambda}_{\bar{s}_j,d}) + \sum_{d=1}^{\mathcal{D}_{j+1}} \Gamma_{2(\bar{s}_{j+1}-m)}^{(\bar{h}_j)}(\lambda_{m,k} - \tilde{\lambda}_{\bar{s}_{j+1},d}) = \sum_{r \in \mathcal{R}_j} \sum_{\ell=1}^{\tilde{\nu}_r} F_2^{(r,m)}(\lambda_{m,k} - \lambda_{r,\ell}) \quad (7.10)$$

where $\bar{h}_j = \frac{\pi}{2(\bar{s}_{j+1}-\bar{s}_j)}$ and, when $m, r \in \mathcal{R}_j$

$$F_2^{(r,m)}(\lambda) = \begin{cases} \Gamma_{4m-4\bar{s}_j}^{(\bar{h}_j)}(\lambda) + 2 \sum_{q=1}^{2m-2\bar{s}_j-1} \Gamma_{2q}^{(\bar{h}_j)}(\lambda) & \text{if } m = r \\ \Gamma_{2r+2m-4\bar{s}_j}^{(\bar{h}_j)}(\lambda) + \Gamma_{2|r-m|}^{(\bar{h}_j)}(\lambda) + 2 \sum_{q=|r-m|+1}^{r+m-2\bar{s}_j-1} \Gamma_{2q}^{(\bar{h}_j)}(\lambda) & \text{if } m \neq r \end{cases} \quad (7.11)$$

Equation (7.10), for $m > \bar{s}_{\mathcal{L}}$, may be rewritten as follows (we used the convention $\bar{s}_{\mathcal{L}+1} = \infty$)

$$-2\pi Q_{\bar{m}+\bar{s}_{\mathcal{L}},k} + \sum_{d=1}^{\mathcal{D}_{\mathcal{L}}} \Phi_1^{(\bar{m})}(\lambda_{\bar{m}+\bar{s}_{\mathcal{L}},k} - \tilde{\lambda}_{\mathcal{L},d}) = \sum_{r \in \frac{1}{2}\mathbb{Z}_{>0}} \sum_{\ell=1}^{\tilde{\nu}_{r+\bar{s}_{\mathcal{L}},k}} \Phi_2^{(r,\bar{m})}(\lambda_{\bar{m}+\bar{s}_{\mathcal{L}},k} - \lambda_{r+\bar{s}_{\mathcal{L}},\ell}) \quad (7.12)$$

for $\bar{m} = 1/2, 1, 3/2, \dots$

Remark 7.2 *Comparing with (4.9), we deduce that the Bethe roots $\lambda_{\bar{m}+\bar{s}_{\mathcal{L}}}$ for the new strings (of length strictly greater than $2\bar{s}_{\mathcal{L}}$) satisfy the Bethe equation of the center of $2m$ -strings for an auxiliary spin chain. More precisely, this auxiliary spin chain is an homogeneous spin $\frac{1}{2}$ chain with $\mathcal{D}_{\mathcal{L}}$ sites and inhomogeneity parameter $\tilde{\lambda}_{\mathcal{L},d}$ at site d .*

This remark, together with the degeneracy of the states $Z_{\mathcal{L}}$ (see (6.11)), lead us to interpret the factors in (7.6) containing $\bar{h}_{\mathcal{L}}$, which are proportional to

$$\prod_{q=1}^{\mathcal{D}_{\mathcal{L}}} \mathcal{K}_1^{(0)}(\tilde{\lambda}_{\mathcal{L},q} - \tilde{\lambda}_{\mathcal{L},j}) \prod_{m \in \{1/2, 1, 3/2, \dots\}} \prod_{\ell=1}^{\tilde{\nu}_{m+\bar{s}_{\mathcal{L}}}} e_{2m}(\tilde{\lambda}_{\mathcal{L},j} - \lambda_{m+\bar{s}_{\mathcal{L}},\ell}), \quad (7.13)$$

as eigenvalues of the transfer matrix of an auxiliary XXX spin $\frac{1}{2}$ chain with the spectral parameter taken at $\tilde{\lambda}_{\mathcal{L},j}$. In fact, all the argumentation do NOT depend on the value of $\bar{s}_{\mathcal{L}}$, nor on all the other values present in the spin chain $(\bar{s}_1, \dots, \bar{s}_{\mathcal{L}-1})$. Therefore, the conclusions obtained for these factors of the S-matrix are completely similar to the usual case of the homogeneous spin $\frac{1}{2}$ chain treated in [22]. We come back on this point in the following section 7.3.

As for the homogeneous spin chain with spin greater than 1, new features appear due to the presence of strings of length smaller than $2\bar{s}_{\mathcal{L}}$. To study the influence of these strings, we must study relation (7.10), for $m \in \mathcal{R}_j$ ($0 \leq j < \mathcal{L}$). Replacing the indices m by $\bar{s}_{j+1} - m$, we get an equivalent relation, for $m = \frac{1}{2}, 1, \dots, \bar{s}_{j+1} - \bar{s}_j - \frac{1}{2}$ and for $k = 1, 2, \dots, \tilde{\nu}_{\bar{s}_{j+1}-m}$

$$\begin{aligned} -2\pi Q_{\bar{s}_{j+1}-m,k} &+ \sum_{d=1}^{\mathcal{D}_j} \Gamma_{2\bar{s}_{j+1}-2\bar{s}_j-2m}^{(\hbar_j)} (\lambda_{\bar{s}_{j+1}-m,k} - \tilde{\lambda}_{\bar{s}_j,d}) + \sum_{d=1}^{\mathcal{D}_{j+1}} \Gamma_{2m}^{(\hbar_j)} (\lambda_{\bar{s}_{j+1}-m,k} - \tilde{\lambda}_{\bar{s}_{j+1},d}) \\ &= \sum_{r \in \{\frac{1}{2}, 1, \dots, \bar{s}_{j+1} - \bar{s}_j - \frac{1}{2}\}} \sum_{\ell=1}^{\tilde{\nu}_{\bar{s}_{j+1}-r}} F_2^{(\bar{s}_{j+1}-r, \bar{s}_{j+1}-m)} (\lambda_{\bar{s}_{j+1}-m,k} - \lambda_{\bar{s}_{j+1}-r,\ell}) \end{aligned} \quad (7.14)$$

Remark 7.3 One recognizes in this relation the Bethe equations within the string hypothesis for an XXZ spin chain with the deformation parameter at root of unity, $q = e^{i\hbar_j} = \exp(i\frac{\pi}{2(\bar{s}_{j+1}-\bar{s}_j)})$, and two types of spins: $\frac{2\bar{s}_{j+1}-2\bar{s}_j-1}{2}$ and $\frac{1}{2}$. Indeed, considering an XXZ spin chain with \mathcal{D}_j sites of spin $\frac{2\bar{s}_{j+1}-2\bar{s}_j-1}{2}$ and inhomogeneity parameters $\tilde{\lambda}_{j,d}$ together with \mathcal{D}_{j+1} sites of spin $\frac{1}{2}$ and inhomogeneity parameters $\tilde{\lambda}_{j+1,d}$, one is led to the following Bethe equations

$$\begin{aligned} &\prod_{d=1}^{\mathcal{D}_j} \frac{\sinh(\hbar_j(x_m - \tilde{\lambda}_{j,d} + i\frac{2\bar{s}_{j+1}-2\bar{s}_j-1}{2}))}{\sinh(\hbar_j(x_m - \tilde{\lambda}_{j,d} - i\frac{2\bar{s}_{j+1}-2\bar{s}_j-1}{2}))} \prod_{d=1}^{\mathcal{D}_{j+1}} \frac{\sinh(\hbar_j(x_m - \tilde{\lambda}_{j+1,d} + \frac{i}{2}))}{\sinh(\hbar_j(x_m - \tilde{\lambda}_{j+1,d} - \frac{i}{2}))} \\ &= \prod_{\ell=1, \ell \neq m}^M \frac{\sinh(\hbar_j(x_m - x_\ell + i))}{\sinh(\hbar_j(x_m - x_\ell - i))} \end{aligned} \quad (7.15)$$

As it has been discussed in [38, 39], the string content of the XXZ chain with q root of the unity in the $\mathcal{D}_j, \mathcal{D}_{j+1} \rightarrow \infty$ limit consists in $2r$ -strings with the restriction $2r < 2(\bar{s}_{j+1} - \bar{s}_j)$ and roots with imaginary part $\bar{s}_{j+1} - \bar{s}_j$. To identify (7.14) with the Bethe equations for the center of $2r$ -strings, one must consider $\lambda_{\bar{s}_{j+1}-r,k}$ has the center of the $2r$ -strings and discard roots with imaginary part. We will call it a restricted string hypothesis.

There are two different S-matrices which depend on the parameters \hbar_j : $S_{j+1,d}$ and $S_{j,d}$. The factor in $S_{j+1,d}$ is proportional to

$$\prod_{q=1}^{\mathcal{D}_j} \mathcal{K}_{\frac{\pi}{\hbar_j}-1}^{(\hbar_j)} (\tilde{\lambda}_{j,q} - \tilde{\lambda}_{j+1,d}) \prod_{q=1}^{\mathcal{D}_{j+1}} \mathcal{K}_1^{(\hbar_j)} (\tilde{\lambda}_{j+1,q} - \tilde{\lambda}_{j+1,d}) \prod_{m \in \mathcal{R}_j} \prod_{\ell=1}^{\tilde{\nu}_m} G_{2(\bar{s}_{j+1}-m)}^{(\hbar_j)} (\tilde{\lambda}_{j+1,d} - \lambda_{m,\ell}) \quad (7.16)$$

This value is similar to the eigenvalue of the transfer matrix of the XXZ model introduced in remark 7.3 inside the restricted string hypothesis. Comparing with the known results of the XXZ model, it is more precisely the transfer matrix with the auxiliary space in the spin $\frac{1}{2}$ representation.

The factor in $S_{j,d}$ is proportional to

$$\prod_{q=1}^{\mathcal{D}_j} \mathcal{K}_1^{(\hbar_j)}(\tilde{\lambda}_{j,q} - \tilde{\lambda}_{j,d}) \prod_{q=1}^{\mathcal{D}_{j+1}} \mathcal{K}_{\frac{\pi}{\hbar_j}-1}^{(\hbar_j)}(\tilde{\lambda}_{j+1,q} - \tilde{\lambda}_{j,d}) \prod_{m \in \mathcal{R}_j} \prod_{\ell=1}^{\tilde{\nu}_m} G_{2(m-\bar{s}_j)}^{(\hbar_j)}(\tilde{\lambda}_{j,d} - \lambda_{m,\ell}) \quad (7.17)$$

Again, this value is similar to the eigenvalue of the transfer matrix of the XXZ model introduced in remark 7.3 but now with the auxiliary space in the spin $\frac{2\bar{s}_{j+1}-2\bar{s}_j-1}{2}$ representation.

In addition, the constraint on the type of strings present in the model suggests that the underlying model is not strictly a XXZ spin chain but rather an RSOS model [40, 41] (see [36] for the analysis) which is in agreement with the counting of states (see section 6.4). Indeed, this general case is very similar to a homogeneous spin $(\bar{s}_{j+1} - \bar{s}_j)$ chain, as treated in [24] where the underlying RSOS structure has been discovered. The only difference lies on the first factor of the L.H.S. of Bethe equations (7.15) which may be explained by replacing a homogeneous RSOS model by a RSOS model with \mathcal{D}_j sites with a jump $2\bar{s}_{j+1} - 2\bar{s}_j - 1$ then \mathcal{D}_{j+1} sites with a jump 1. This interpretation justifies the choice done at the end of section 6.4 between the different RSOS model that are allowed when one looks only at the number of states.

7.3 Conjecture for the scattering matrix

All the considerations of previous subsections 7.1 and 7.2 allow us to propose an educated guess for the scattering matrix of the model. This S-matrix depends on the type of particle we consider: the particles of type \mathcal{L} must be treated separately from the particles of type j ($1 \leq j < \mathcal{L}$).

7.3.1 Scattering of type \mathcal{L} particles

The particles of type \mathcal{L} scatter non trivially only with particles of the same type and with particles of type $\mathcal{L} - 1$ (if this type exists). As explained before, this type of particle has a spin $\frac{1}{2}$ as well as a supplementary degree of freedom satisfying a RSOS model. The non trivial part of the S-matrix for the particle of type \mathcal{L} acts only on

$$\mathcal{H}^{RSOS}(\mathcal{D}_{\mathcal{L}-1}; \mathcal{D}_{\mathcal{L}}; \bar{s}_{\mathcal{L}} - \bar{s}_{\mathcal{L}-1}) \otimes (\mathbb{C}^2)^{\otimes \mathcal{D}_{\mathcal{L}}} \quad (7.18)$$

We remind (see section 7.2) that the equations (7.12) and (7.13) allowing one to compute the part of the scattering matrix acting on $(\mathbb{C}^2)^{\otimes \mathcal{D}_{\mathcal{L}}}$ do not depend on the values of $\bar{s}_1, \dots, \bar{s}_{\mathcal{L}}$. Therefore, the spin part of the scattering matrix for the d^{th} particle of type \mathcal{L} is similar to the one of the usual homogeneous spin $\frac{1}{2}$ chain [22]. As usual, in order to write the scattering matrix, we introduce the following transfer matrix

$$t^{(\mathcal{L})}(\lambda) = \text{tr}_0 S_{01}(\lambda - \tilde{\lambda}_{\mathcal{L},1}) \dots S_{0\mathcal{L}}(\lambda - \tilde{\lambda}_{\mathcal{L},\mathcal{D}_{\mathcal{L}}}) \quad (7.19)$$

where

$$S(\lambda) = \mathcal{K}_1^{(0)}(\lambda) R(-\lambda), \quad (7.20)$$

and we have used the R-matrix $R(\lambda)$ defined by (3.4).

For the RSOS part of the scattering matrix, we use the results of [24]. This RSOS part acts on $\mathcal{H}^{RSOS}(\mathcal{D}_{\mathcal{L}-1}; \mathcal{D}_{\mathcal{L}}; \bar{s}_{\mathcal{L}} - \bar{s}_{\mathcal{L}-1})$. Let us define the Boltzmann weights of the usual RSOS model [40, 41] as well as the ones of the fused RSOS model [42, 36] (see also [43])

$$W_{\hbar} \left(\frac{d}{a} \middle| \frac{c}{b} \middle| \lambda \right) = \delta_{ac} - (-1)^{\frac{a-c}{2}} \frac{\sinh(\hbar\lambda)}{\sinh(\hbar(\lambda-i))} \sqrt{\frac{\sin(\hbar(a+1)) \sin(\hbar(c+1))}{\sin(\hbar(b+1)) \sin(\hbar(d+1))}} \delta_{bd}, \quad (7.21)$$

$$W_{\hbar} \left(\frac{b_1}{a_1} \parallel \frac{b_{2s}}{a_{2s}} \middle| \lambda \right) = \sum_{a_2, \dots, a_{2s-1}} \prod_{n=1}^{2s-1} W_{\hbar} \left(\frac{b_n}{a_n} \middle| \frac{b_{n+1}}{a_{n+1}} \middle| \lambda + i(n-2s+1) \right), \quad (7.22)$$

with $\hbar = \frac{\pi}{2s+2}$ and $2s+2$ the restriction parameter of the RSOS model considered. The simple lines between the indices in the notation for the Boltzmann weights means that there is a jump of 1 between these ones whereas the double lines means that the jump is $2s-1$. Let us remark that R.H.S. of (7.22) defining the fused Boltzmann weights are well-defined only for $s > \frac{1}{2}$. For $s = \frac{1}{2}$, the fused Boltzmann weight are 1.

We introduced the RSOS transfer matrix by its entries, for $0 \leq j < \mathcal{L}$ and $1 \leq d \leq \mathcal{D}_{j+1}$,

$$\begin{aligned} < E(\{a'\}_{\mathcal{D}_j}; \{b'\}_{\mathcal{D}_{j+1}}) | t_1^{(j,d)}(\lambda) | E(\{a\}_{\mathcal{D}_j}; \{b\}_{\mathcal{D}_{j+1}}) > = \mathcal{N}^{(j)}(\lambda) \prod_{q=1}^{\mathcal{D}_j} W_{\hbar'_j} \left(\frac{a_{q-1}}{a'_q} \parallel \frac{a_q}{a'_{q+1}} \middle| \lambda - \tilde{\lambda}_{j,q} \right) \\ & \times \prod_{q=1}^{d-1} W_{\hbar'_j} \left(\frac{b_{q-1}}{b'_q} \middle| \frac{b_q}{b'_{q+1}} \middle| \lambda - \tilde{\lambda}_{j+1,q} \right) \prod_{q=d}^{\mathcal{D}_{j+1}-1} W_{\hbar'_j} \left(\frac{b_{q-1}}{b'_q} \middle| \frac{b_q}{b'_{q+1}} \middle| \lambda - \tilde{\lambda}_{j+1,q+1} \right) \end{aligned} \quad (7.23)$$

where E has been defined by (6.26), $\hbar'_j = \frac{\pi}{2\bar{s}_{j+1}-2\bar{s}_j+2}$, $a'_{\mathcal{D}_{j+1}} = b'_1$ (by convention) and the normalization

$$\mathcal{N}^{(j)}(\lambda) = \prod_{q=1}^{\mathcal{D}_j} \mathcal{K}_{2\bar{s}_{j+1}-2\bar{s}_j-1}^{(\hbar'_j)}(\tilde{\lambda}_{j,q} - \lambda) \prod_{q=1}^{\mathcal{D}_{j+1}} \mathcal{K}_1^{(\hbar'_j)}(\tilde{\lambda}_{j+1,q} - \lambda). \quad (7.24)$$

All these considerations lead to the following conjecture:

Conjecture 7.1 *The scattering matrix of the d^{th} particle of type \mathcal{L} takes the form*

$$S_{\mathcal{L},d} \sim t^{(\mathcal{L})}(\tilde{\lambda}_{\mathcal{L},d}) t_1^{(\mathcal{L}-1,d)}(\tilde{\lambda}_{\mathcal{L},d}) \quad (7.25)$$

where \sim stands for 'equals up to a conjugation'. In (7.25), $t^{(\mathcal{L})}(\tilde{\lambda}_{\mathcal{L},d})$, acting on $(\mathbb{C}^2)^{\otimes \mathcal{D}_{\mathcal{L}}}$, is the transfer matrix (7.19) taken at the value $\lambda = \tilde{\lambda}_{\mathcal{L},d}$ and $t_1^{(\mathcal{L}-1,d)}(\tilde{\lambda}_{\mathcal{L},d})$, acting on $\mathcal{H}^{RSOS}(\mathcal{D}_{\mathcal{L}-1}; \mathcal{D}_{\mathcal{L}}; \bar{s}_{\mathcal{L}} - \bar{s}_{\mathcal{L}-1})$, the RSOS transfer matrix (7.23) taken at the value $\lambda = \tilde{\lambda}_{\mathcal{L},d}$.

This conjecture allows us to reproduce the scattering matrices obtained by [24, 6] for the homogeneous spin s chain by putting $\mathcal{L} = 1$ (then $\mathcal{D}_{\mathcal{L}-1} = \mathcal{D}_0 = 0$), $\bar{s}_{\mathcal{L}} - \bar{s}_{\mathcal{L}-1} = s$ and $\hbar'_{\mathcal{L}-1} = \frac{\pi}{2s+2}$. Obviously, we recover also the result for the spin $\frac{1}{2}$ chain obtained previously in [22] which is the previous case for $s = \frac{1}{2}$. In this case, the transfer matrix $t_1^{(0,d)}(\lambda)$ must be equal to 1. To show that, we used the following relations

$$\mathcal{K}_1^{(\pi/3)}(-x) W_{\pi/3} \left(\frac{0}{1} \middle| \frac{1}{0} \middle| x \right) = 1 \quad \text{and} \quad \mathcal{K}_1^{(\pi/3)}(-x) W_{\pi/3} \left(\frac{1}{0} \middle| \frac{0}{1} \middle| x \right) = 1. \quad (7.26)$$

In the case where $\bar{s}_{\mathcal{L}} - \bar{s}_{\mathcal{L}-1} = 1/2$, the RSOS becomes trivial (its dimension is one) and the transfer matrix is reduced to a scalar function

$$t_1^{(\mathcal{L}-1,d)}(\tilde{\lambda}_{\mathcal{L},d}) = \prod_{q=1}^{\mathcal{D}_{\mathcal{L}-1}} i \coth \left(\frac{\pi}{2} \left(\tilde{\lambda}_{\mathcal{L},d} - \tilde{\lambda}_{\mathcal{L}-1,q} + \frac{i}{2} \right) \right). \quad (7.27)$$

We have used relations (7.26) and (2.9). In particular, we recognize the results obtained in [25, 26] where the alternating spin $(\frac{1}{2}, 1)$ chain is treated.

7.3.2 Scattering for particles of type $1, 2, \dots, \mathcal{L} - 1$

The particles of type j ($1 \leq j < \mathcal{L}$) scatter non trivially only with particles of type $j-1$, j and $j+1$. The non trivial part of the S-matrix for the particle of type j acts only on

$$\mathcal{H}^{RSOS}(\mathcal{D}_{j-1}; \mathcal{D}_j; \bar{s}_j - \bar{s}_{j-1}) \otimes \mathcal{H}^{RSOS}(\mathcal{D}_j; \mathcal{D}_{j+1}; \bar{s}_{j+1} - \bar{s}_j). \quad (7.28)$$

The scattering matrix acting on the first space introduced above is very similar to the one computed in the previous section and is related to the transfer matrix (7.23).

For the scattering matrix acting on the second space, as suggested by relation (7.17), we must introduce a transfer matrix with a fused auxiliary space using the following Boltzmann weights $W_h \left(\frac{d}{a} \middle| \frac{c}{b} \middle| \lambda \right)$ and $W_{\bar{h}} \left(\frac{d}{a} \middle| \frac{c}{b} \middle| \lambda \right)$. However, in [36], it is shown there exists a gauge transformation between the Boltzmann weights with a fused auxiliary space and the usual ones. Therefore, we introduce the following transfer matrix, for $1 \leq j < \mathcal{L}$ and $1 \leq d \leq \mathcal{D}_j$,

$$\begin{aligned} \langle E(\{a'\}_{\mathcal{D}_j}; \{b'\}_{\mathcal{D}_{j+1}}) | t_{2s-1}^{(j,d)}(\lambda) | E(\{a\}_{\mathcal{D}_j}; \{b\}_{\mathcal{D}_{j+1}}) \rangle &= \mathcal{M}^{(j)}(\lambda) \prod_{q=1}^{\mathcal{D}_{j+1}} W_{h'_j} \left(\frac{b_{q-2}}{\bar{b}_{q-1}} \middle| \frac{\bar{b}_{q-1}}{b'_q} \middle| \lambda - \tilde{\lambda}_{j+1,q} \right) \\ &\times \prod_{q=1}^{d-1} W_{h'_j} \left(\frac{a_{q-1}}{\bar{a}'_q} \middle| \frac{\bar{a}_q}{a'_{q+1}} \middle| \lambda - \tilde{\lambda}_{j,q} \right) \prod_{q=d}^{\mathcal{D}_j-1} W_{h'_j} \left(\frac{a_{q-1}}{\bar{a}'_q} \middle| \frac{\bar{a}_q}{a'_{q+1}} \middle| \lambda - \tilde{\lambda}_{j,q+1} \right) \end{aligned} \quad (7.29)$$

where $h'_j = \frac{\pi}{2\bar{s}_{j+1}-2\bar{s}_j+2}$, $b_{-1} = a_{\mathcal{D}_j-1}$, the overlined indices $\bar{a} = 2\bar{s}_{j+1} - 2\bar{s}_j - a$ and the normalization

$$\mathcal{M}^{(j)}(\lambda) = \prod_{q=1}^{\mathcal{D}_j} \mathcal{K}_1^{(h'_j)}(\tilde{\lambda}_{j,q} - \lambda) \prod_{q=1}^{\mathcal{D}_{j+1}} \mathcal{K}_{2\bar{s}_{j+1}-2\bar{s}_j-1}^{(h'_j)}(\tilde{\lambda}_{j+1,q} - \lambda). \quad (7.30)$$

After these definitions, we can give the conjectured form of the scattering matrix:

Conjecture 7.2 *The scattering matrix of the d^{th} particle of type j can be written as follows*

$$S_{j,d} \sim t_1^{(j-1,d)}(\tilde{\lambda}_{j,d}) t_{2(\bar{s}_{j+1}-\bar{s}_j)-1}^{(j,d)}(\tilde{\lambda}_{j,d}). \quad (7.31)$$

With this conjecture, the scattering matrix $S_{1,d}$ of the alternating spin $(\frac{1}{2}, 1)$ chain reduces to

$$S_{1,d} \sim \prod_{q=1}^{\mathcal{D}_2} i \coth \left(\frac{\pi}{2} \left(\tilde{\lambda}_{1,d} - \tilde{\lambda}_{2,q} + \frac{i}{2} \right) \right). \quad (7.32)$$

This result is consistent with the previous result (7.27) (the scattering of the particle 1 on particle 2 must be the same than the scattering of 2 on 1). It reproduces also the results of [25, 26] and, in particular, it shows that the particles of type 1 scatter trivially.

Finally, to support this conjecture and the choice of RSOS models, we can look for the central charge of the underlying conformal model computed previously in [44]

$$c = \mathcal{L} + \sum_{j=1}^{\mathcal{L}} \left(2 - \frac{3}{\bar{s}_j - \bar{s}_{j-1} + 1} \right). \quad (7.33)$$

We recognize in each term $(2 - \frac{3}{\bar{s}_j - \bar{s}_{j-1} + 1})$ the central charge of a RSOS model with the restriction parameter $2\bar{s}_j - 2\bar{s}_{j-1} + 2$ (see [36]).

7.3.3 Scattering for 2 particles

Let us remark that for $\mathcal{D}_j = 2$ and $\mathcal{D}_k = 0$ ($k \neq j$), it is possible to solve the Bethe equations of section 7.2 and to compute exactly a 2-particle scattering matrix thanks to the results of section 7.1. However, the results computed in this way disagree, in general, with the results obtained via the conjectures (7.25) or (7.31). This discrepancy appeared already in the case of homogeneous spin s chain with $s > \frac{1}{2}$: the 2-body scattering matrices computed in [23] are different from the one computed in [24] (whereas both results agreed for $s = \frac{1}{2}$). This inconsistency has been attributed in [25, 26] to the non-validity of the string hypothesis (see also the remark 4.1). To support this point, we emphasize that the computation of the central charge has the same feature. Indeed, this computation using the Bethe equations inside the string hypothesis provides $c = 1$ and is different from the one obtained by thermodynamical considerations [45] or by numerical investigations of the Bethe equations without string hypothesis [46] which give $c = \frac{3s}{s+1}$ (for $s = \frac{1}{2}$, both results are again in agreement). Finally, let us remark that this disagreement occurs when the RSOS structure becomes non trivial (i.e. when the RSOS space becomes strictly greater than one).

For the more general case of L_0 -regular spin chains treated in this paper, we have assumed, in order to guess the conjectures, that similar discrepancies in the computation of the scattering matrix acting non trivially on $\mathcal{H}^{RSOS}(\mathcal{D}_j; \mathcal{D}_{j+1}; \bar{s}_{j+1} - \bar{s}_j)$ take place also in the cases when $\bar{s}_{j+1} - \bar{s}_j > \frac{1}{2}$ (i.e. when the corresponding RSOS space becomes non trivial) and only in these cases. As for the homogeneous spin chain, this is corroborated by the computation of the central charge inside the string hypothesis and by comparing it with (7.33). Indeed, generalizing the computations done for example in [32, 33, 47], we proved that the central charge is equal to \mathcal{L} inside the string hypothesis. Therefore, to get the value of the central charge given in (7.33), one must add a non-vanishing term to \mathcal{L} each time $s_{j+1} - \bar{s}_j > \frac{1}{2}$.

From the above discussions, it seems as though the computations done in subsections 7.1 and 7.2 are useless. Nevertheless, let us point out the two following points. Firstly, thanks to the comparison with the homogeneous spin chain results [24], they allow us to give an educated guess for the scattering matrices. Secondly, as argued in [25], the scattering matrices obtained inside the string hypothesis (i.e. the ones of subsections 7.1 and 7.2) may be also the ones of an underlying quantum field theory. However, this theory would be obtained as a limit when one sends to zero

first the temperature then the magnetic field, whereas the theory corresponding to our conjectured scattering matrices would be obtained when one sends to zero first the magnetic field and then the temperature.

8 Conclusion: open problems

For a general L_0 -regular closed XXX spin chain, we have identified the elementary excitations of the chain. They consist in spin-1/2 spinons (associated to the highest spin sites entering the chain) and scalar particles, whose different types are related to the other different sites in the chain. Then, we have conjectured the general form of the scattering matrix for these elementary excitations. It makes appear generalized RSOS models, and we have given the corresponding Boltzmann weights. The first question to address is obviously about the validity of this conjecture. We have argued about it by computing the central charge of the associated conformal models, but a complete proof is still lacking.

It is also natural to ask whether this approach can be generalized to other algebras or superalgebras. Indeed, a first account on L_0 -regular spin chains based on $gl(N)$ can be found in [18, 21]. The relevance of such general integrable spin chains in condensed matter physics have been pointed out in e.g. [48]. However, the calculation of the scattering matrix has been done only for homogeneous spin chain with particular representations [49, 50]. A general treatment remains to be done for these models. This open problem is very promising, since it could be linked to RSOS models based on $gl(N)$.

The situation is very similar when one considers deformations (quantum groups) and/or superalgebras $gl(M|N)$. For the quantum groups $\mathcal{U}_q(gl_N)$ (and in particular $\mathcal{U}_q(gl_2)$, related to XXZ chain), the same algebraic approach to construct integrable L_0 -regular spin chain can be done (see first accounts in [51, 52]). The excited states for homogeneous arbitrary spin chain based on $\mathcal{U}_q(gl_2)$ and their scattering matrix have been studied in [30]: they also show internal degrees of freedom. The general case (based on $\mathcal{U}_q(gl_N)$) remains to be done. For superalgebras, again the super-Yangian $Y(gl(M|N))$ can be investigated through the same method (see e.g. [53]) and should lead to generalized super-RSOS models. Finally, $\mathcal{U}_q(gl(M|N))$ can also be treated in the same way, see for instance [52] where the nested Bethe ansatz is done in a unified way for all these cases ($Y(gl_N)$, $\mathcal{U}_q(gl_N)$, $Y(gl(M|N))$ and $\mathcal{U}_q(gl(M|N))$) and at the algebraic level. The computation of the scattering matrices is still an open problem.

For other (orthogonal, symplectic or exceptional) algebras or the orthosymplectic superalgebra, the situation is different. Indeed, the whole construction of ‘algebraic spin chains’ (as introduced in [18]) relies on the so-called evaluation morphism between the Yangian $Y(gl_N)$ and the envelopping algebra $\mathcal{U}(gl_N)$. This morphism does not exist for these other classical algebras, so that one needs to work at the level of representations directly. In fact, the scattering matrices has been computed in few cases (for example, in [54], for the homogeneous $osp(1|n)$ spin chain in the fundamental representation). This indicates that a general method based on a different approach should exist, but it is not known up to now.

Another question that rises is the case of open spin chains. Indeed, in [20] and [53] these chains have also been treated, and the nested Bethe ansatz for ‘algebraic open spin chains’ based on $Y(gl_N)$, $\mathcal{U}_q(gl_N)$, $Y(gl(M|N))$ and $\mathcal{U}_q(gl(M|N))$ can be found in [55]. Thus, we expect that the procedure can be applied to these cases too. Scattering matrices for models with boundaries when the spins are in the fundamental representation have been computed in [50, 56, 54].

Finally, let us note that one could use these generalized spin chains to define new integrable t-J models with impurities in the spirit of [57], using a Jordan-Wigner type transformation.

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